EFI 72-39

NAL-THY-82

Nov. 1972

# REGGE BEHAVIOR AND FIXED ANGLE SCATTERING*

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*Work supported in part by the National Science Foundation: contract No. NSF GP 32904X; and by the Universities Research Association, Inc., under contract with the United States Atomic Energy Commission.

ABSTRACT

It is shown that, if one expresses the hadronic scattering amplitude as a sum of contributions from individual quark diagrams, each of these quark diagram contributions will obey a logarithmic scaling law in the high energy fixed angle limit. The ingredients of the proof are the linear motion of singularities in the complex angular momentum plane and certain analyticity assumptions. The logarithmic scaling law emerges from the requirement of consistency of the behaviors of the individual quark diagram amplitude in the fixed angle high energy regime and in the Regge regime. Each quark diagram is found to have an intrinsic scale set by the slope of the "dominant" singularity in the complex angular momentum plane of any one of its channels, i.e., the singularity whose trajectory lies highest for large negative values of that channel's Mandelstam variable. Complex angular momentum plane singularities that are dominant in different channels of the same quark diagram must therefore have the same slope. As the Pomeranchuk singularity and Regge-Regge cuts are dominant in different channels of the same (twisted loop) quark diagram, it follows that they must have the same slope of $\sim 0.42\ \mathrm{GeV}^{-2}$. Assuming a small number of quark diagrams to dominate the physical scattering amplitude at present energies and fixed angles, the differential cross-section is found to also obey a logarithmic scaling law which agrees with the wide-angle pp scattering data.

# 1. INTRODUCTION

Hadronic two body reactions have been extensively studied in the Regge regime in which one of the Mandelstam variables (s, for instance) becomes large, while another (t or u) is kept fixed. Much less attention has been devoted so far, to the regime of high energy fixed angle scattering. In this regime all three Mandelstam variables ($|s|$, $|t|$ and $|u|$) become large, but all their ratios are kept fixed. In other words, kinematic variables with dimensions of a mass squared become large and only dimensionless quantities (like $\cos\theta$) are finite. If all hadrons had masses smaller than some finite mass M, one might expect that in this regime masses become irrelevant and amplitudes exhibit simple scaling properties. Proposals of this type have been made by a number of authors.[1] They all claim that there exists a dimension d, dependent on the dynamical imput, such that $s^d A(s,\theta)$ (A = reaction amplitude) scales (i.e., depends only on the scattering angle $\theta$) at high energies:

$$s^d A(s,\theta) \xrightarrow[\theta = \text{fixed}]{|s| \to \infty} f(\theta) \tag{1.1}$$

The real hadron spectrum does not exhibit any cut-off mass M. Rather, resonances equally spaced in mass-squared and with ever increasing masses are indicated by experiment. If such an unbounded hadron spectrum is accepted, then a natural scale for fixed angle high energy scattering is set by the spacing of about 1.2 $\text{GeV}^2$ between consecutive (though arbitrarily high lying) resonances. In view of the existence of such an intrinsic scale, one would have to be somewhat skeptical about scaling laws of the type (1.1), and explore possible alternatives. Most importantly, once a scale is provided, an exponential behavior, for instance of the type

$$A(s,\theta) \xrightarrow[\theta = \text{fixed}]{|s| \to \infty} p(s,\theta)\, \exp\left(-\alpha' s\, f(\cos\theta)\right) \tag{1.2}$$

may set in. Here $p(s,\theta)$ is bounded by some power of $s$, $\alpha'$ is the scale parameter mentioned above (slope of Regge trajectories) and $f(\cos\theta)$ some function of the scattering angle. With this possibility in mind, we shall explore fixed angle high energy scattering in the context of a quark model with linearly rising Regge trajectories. Dual resonance models (with loop corrections included) are of this type but our arguments will be more general. Our major result is that the contribution of any given (tree or loop) quark diagram D to the scattering amplitude $A(s,z)$ in the $s \to \infty$, $z = \cos\theta$ = fixed limit obeys the <u>logarithmic scaling law</u>

$$-(\alpha'_D s)^{-1} \ln A(s,z) \xrightarrow[z = \text{fixed}]{|s| \to \infty} f(z) \tag{1.3a}$$

Here the "scale of the quark diagram D", $\alpha'_D$, is set by the slope of the "dominant" singularity in the complex angular momentum plane of any one of the diagram's (non-empty) Mandelstam channels, i.e., the singularity whose trajectory lies highest for large negative values of that channel's Mandelstam variable. The function $f(z)$ takes the <u>diagram-independent</u> (except for details of the imaginary part related to empty channels) form

$$f(z) = \frac{1-z}{2} \ln \frac{2}{1-z} + \frac{1+z}{2} \ln \frac{2}{1+z} \tag{1.3b}$$

first discussed by Veneziano[2] in the special case of tree diagrams. Observe that the logarithmic scaling law (1.3) implies the existence of a <u>unique</u> scale [illegible] each quark diagram. Therefore, the slopes of the dominant Regge singu- [illegible] all (non-empty) Mandelstam channels of a quark diagram must be [illegible]. As a consequence of this result, we shall prove that the Pomeranchuk singularity, responsible for diffraction must have the same slope as Regge-Regge cuts or in other words a slope half that of the usual (e.g. $\rho$ or $\Delta$) Regge trajectories.

The logarithmic scaling law (Eq. (1.3)) emerges from the requirement that the behavior of the individual quark diagram aplitude in the fixed angle high energy regime and in the Regge regime be consistent. Furthermore, we shall see that the logarithmic scaling law agrees with the existing data on wide angle pp-scattering.

The logarithmic scaling law (1.3) leads to the cut-off in transverse momentum often postualted in the phenomenology of hadronic reactions. This cut-off seems to be an inescapable consequence of linearly rising Regge-trajectories.

Logarithmic scaling laws can presumably be generalized to multiparticle exclusive processes and thus, via Mueller arguments, to inclusive processes.*

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*A special logarithmic scaling law for single inclusive processes has [illegible] noted in the context of dual resonance models[3] and favorably compared [illegible]

## 2. HIGH ENERGY FIXED ANGLE SCATTERING IN DUAL RESONANCE MODELS AS A GUIDE.

Dual resonance models have an intrinsic scale: the slope $\alpha'$ of the Regge trajectories. We shall now find some regularities exhibited by high energy fixed angle scattering in dual resonance models. Then we shall abstract these regularities and in the next section prove that they hold on more general grounds.

In dual resonance models, to each quark diagram there corresponds a unique amplitude calculated according to the "Feynman rules" of the model. If one Mandelstam variable (say, s) becomes large while another variable (say, t) is kept fixed, the amplitude for a given quark diagram without loops either exhibits Regge behavior $\beta(t)s^{\alpha(t)}$ with $\alpha(t) = \alpha(0) + \alpha't$ or falls exponentially with s if the t-channel is "empty" (i.e., contains no Regge poles). In the same kinematic limit the amplitude corresponding to the quark diagram with loops D behaves in general like

$$\sum_i \beta_{Dti}(t)\, s^{\alpha_{Dti}(t)} (\ln s)^{\lambda_{Dti}(t)} (\ln\ln s)^{\mu_{Dti}(t)} \cdots\cdots$$

and the singularities in the t-channel angular momentum plane are either multiple poles (in which case one deals with a renormalization effect to the input Regge poles) or branch points (Regge-Regge cut, Pomeranchuk singularity*, Pomeranchuk-Regge cut, etc...) the remarkable feature of all these singularities is that their positions $\ell = \alpha_{Dti}(t)$ in the complex $\ell$-plane depend linearly on t**

$$\alpha_{Dti}(t) = \alpha_{Dti}(0) + \alpha'_{Dti}\, t$$

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*The Pomeranchuk could be a Regge pole, see [5].

**This comes as no surprise as all these singularities are derived in higher orders from linear input trajectories.

Let us order the singularities $\alpha_{Dti}(t)$ according to their slope and intercept

$$\alpha'_{Dt1} \leq \alpha'_{Dt2} \leq \alpha'_{Dt3} \leq$$

and

$$\alpha_{Dtn}(0) \geq \alpha_{Dt\,n+1}(0) \qquad \text{if} \quad \alpha'_{Dtn} = \alpha'_{Dt\,n+1}$$

Then for large negative t, $\alpha_{Dt1}(t)$ will determine the asymptotic behavior of the amplitude. We shall call $\alpha_{Dt1}(t)$ the dominant singularity in the t-channel and denote its trajectory by

$$\alpha_{Dt}(t) = \alpha_{Dt}(0) + \alpha'_{Dt}\, t \tag{2.1}$$

(dropping for simplicity the index 1).

We can summarize all this in the form

A) In the limit $|s| >> |t|$ (or $s \to \infty$, t = fixed and sufficiently negative) the contribution $A_D(s,t)$ of the quark diagram D to the scattering amplitude behaves like $\beta_{Dt}(t)s^{\alpha_{Dt}(t)}$ up to logarithmic factors or in other words

$$\ln A_D(s,t) \xrightarrow[|s|>>|t|]{} \alpha_{Dt}(t) \ln s + \ln \beta_{Dt}(t) \tag{2.2}$$

The Regge trajecories $\alpha_{Dt}(t)$ are linear* in t with slopes depending on the diagram D and a priori also on the choice of the t-channel. In the case of a diagram without loops and "empty" t-channel $A_D(s,t)$ falls exponentially with s.

In dual resonance models, for all diagrams with fewer than two loops (fig. 1) it has been found[4,5] that

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*If one sums classes of quark diagrams, the sum of various linear singularities may add up to one "renormalized" slightly nonlinear pole or branch point. We are talking however of individual quark daigrams, so we need not worry about such "collective" effects.

$$\ln \rho_{Dt}(t) \xrightarrow[|t|\to\infty]{} -\alpha'_{Dt}\, t \ln \alpha'_{Dt}\, t \tag{2.3}$$

This leads us to conjecture that in general

B) the limiting form (2.3) holds for all quark diagrams no matter how many loops they have.

Consider now the limit of high energy fixed angle scattering, i.e.,

$$|s|,\ |t|,\ |u| \to \infty$$

$$\left|\frac{s}{t}\right|,\ \left|\frac{t}{u}\right|,\ \left|\frac{u}{s}\right| = \text{fixed} \tag{2.4a}$$

or equivalently

$$|s| \to \infty$$

$$z = \cos\theta_s \approx 1 + \frac{2t}{s} = \text{fixed} \tag{2.4b}$$

or its crossed $s \leftrightarrow t$, $s \leftrightarrow u$ forms.

Again on the basis that the statements have been verified[2,6] for all diagrams with less than two loops we conjecture that

C) for any quark diagram D, in the limit (2.4), the combination $(\alpha'_D s)^{-1} \ln A_D(s,z)$ scales i.e., is a function $f_D(z)$ only of z.

D) The function $f_D(z)$ is universal i.e., does not depend on the diagram D. We shall therefore call it f(z) from now on.

E) For any quark diagram with or without loops f(z) is given essentially by Veneziano's Born term formula

$$f(z) = \frac{1-z}{2} \ln \frac{2}{1-z} + \frac{1+z}{2} \ln \frac{2}{1+z} \tag{2.5}$$

There is some ambiguity about where the cuts arising from the logarithms should lie in the complex z plane. This question has physical content in terms of the empty channel question and will be studied in detail in Section 4 .

In view of statement C) we shall henceforth refer to the kinematic limit (2.4) as the logarithmic scaling regime or the logarithmic scaling limit.

Obviously, it is not our intention to leave all these statements at the conjectural level. What we shall rather do is, to find the relationships between these statements. Of course, we shall find that they are not independent. Much less than the full machinery of dual resonance models need be activated to arrive at these results. We shall see in Section 4 that statements B)-E) all follow from the assumed Regge behavior with linear trajectories plus a certain analyticity assumption to be formulated in Section 3 when we require consistency between the Regge behavior and the behavior at fixed angle.

## 3. CONSISTENCY REQUIREMENT IN THE "OVERLAP" REGION AND THE ANALYTICITY ASSUMPTION.

The basis of the analysis which follows is the requirement of consistency between the Regge regime and the logarithmic scaling regime in the kinematic region of "overlap" where they both apply. More specifically, every quark diagram amplitude obeys equation (2.3) in the Regge regime (i.e., for $|s| \gg |t|$). On the other hand the logarithmic scaling regime obtains where $|s|$ and $|t|$ are both large but their ratio is fixed. When both $|s|$ and $|t|$ are large ($|s|, |t| \gg m_\rho^2$) and $|s| \gg |t|$, both the Regge and the logarithmic scaling regimes are applicable, (the "overlap" region) and we <u>require</u> that they be mutually consistent.

As can be easily checked using eq. (2.4) the overlap between the logarithmic scaling and the t-channel Regge ($|t| \ll |s|, |u|$) regimes occurs for

$$|s| \to \infty \qquad\qquad z \to +1 \tag{3.1a}$$

Similarly, overlap with the u- and s- channel Regge regimes corresponds to

$$|s| \to \infty \qquad\qquad z \to -1 \tag{3.1b}$$

and

$$|s| \to \infty \qquad\qquad z \to -\infty \tag{3.1c}$$

respectively.

Thus enforcing the consistency criteria in the overlap region is going to give us information on $(\alpha'_{Di}s)^{-1} \ln A_D(s,z)$ in the kinematic regions of (3.1). In order to determine the behavior of this function in the entire z plane for $|s| \to \infty$, some kind of analyticity assumption is essential. We again look to the dual resonance model for guidance. In the dual resonance model we know that for quark diagrams with fewer than two loops

$$(\alpha'_{Dt}\, s)^{-1} \ln A_D(s,t) \xrightarrow[z\ \text{fixed}]{|s|\to\infty} f(z) \tag{3.2}$$

where f(z) is given essentially by eq. (2.5). We observe two crucial properties of f(z). First f(z) is an analytic function of z whose only singularities are located at z = ± 1 and z = ∞. Recall that in the dual resonance model (again for quark diagrams with fewer than two loops) Regge behavior (2.2) occurs and the Regge residues obey eq. (2.3). This Regge behavior, in the overlap region (3.1a), can be written in the form

$$-(\alpha'_{Dt}\, s)^{-1} \ln A_D(s,z) \underset{\substack{|s|\to\infty \\ z\to 1}}{\approx} \frac{1-z}{2} \ln \frac{2}{1-z} \tag{3.3a}$$

where we are neglecting terms which are both zero and regular at z = 1. Likewise we have

$$-(\alpha'_{Dt}\, s)^{-1} \ln A_D(s,z) \underset{\substack{|s|\to\infty \\ z\to -1}}{\approx} \frac{\alpha'_{Du}}{\alpha'_{Dt}} \frac{1+z}{2} \ln \frac{2}{1+z} \tag{3.3b}$$

and

$$-(\alpha'_{Dt}\, s)^{-1} \ln A_D(s,z) \underset{\substack{|s|\to\infty \\ z\to -\infty}}{\approx} -\frac{\alpha'_{Ds}}{\alpha'_{Dt}} \ln z \tag{3.3c}$$

Limiting behavior implied by the consistency conditions eqs. (3.3) is, of course, exhibited by f(z) of eq. (2.5).

The second important property of the function f(z) is that not only does f(z) satisfy eqs. (3.3) but these condtions are sufficient to determine its form. More explicitly, if we require that the nature of the singularities of f(z) at z = ± 1 and z = ∞ be of exactly the form given by the consistency criteria (3.3) (e.g., that there be no term $(1-z)^2 \ln(1-z)$ in $(\alpha'_{Dt} s)^{-1} \ln A_D(s,t)$ as z → 1) then f(z) _must_ have the general form (2.5). With the above discussion as a guide we state our analyticity assumption in the form:

F) The leading term in the asymptotic (large s) expansion of the function $(\alpha'_{Dt}s)^{-1}\ln A_D(s,t)$ is analytic in z except for possible singularities at $z = \pm 1$ and $z = \infty$. The nature of these singularities is completely determined by the consistency criteria in the overlap region.

This analyticity assumption may not seem natural at first sight and we shall therefore try to motivate it further. A priori, the function $(\alpha'_{Dt}s)^{-1}$ $\ln A_D(s,t)$ could be expected to have many more singularities in the variable z than those discussed above. In the limit $s \to \infty$, $A_D(s,z)$ has singularities at the points

$$z_{n+} \simeq \left(1 + \frac{2m_{tn}^2}{s}\right) \quad , \quad z_{n-} = -\left(1 + \frac{2m_{un}^2}{s}\right) \tag{3.4}$$

Here $m_{tn}$ and $m_{un}$ are the masses of higher lying particles and/or thresholds in the t and u channels respectively. Because of the linearity of the trajectories, even in zero loop diagrams (for which no multi-particles thresholds exists), there will exist $m_n$'s larger than any mass M. Moreover, since the difference $m_{n+1} - m_n^2$ is constant ( $\sim 1.2\ \mathrm{GeV}^2$), one sees that as $s \to \infty$ the singularities (3.4) cover even more densely the half-infinite straight lines

$$z = 1 + \rho e^{-i\varphi}$$

and $\qquad\qquad\qquad\qquad \rho = \mathrm{Re}\,\rho > 0 \qquad\qquad (3.5)$

$$z = -1 - \rho e^{-i\varphi}$$

(here $\varphi = \arg s$). If, for example, all the original singularities were poles, the $s \to \infty$ limit yields two continuous lines of poles extending from $+1$ to $\infty$ and from $-1$ to $-\infty$. This implies two cuts with branch points at $z=\pm1$ and $\infty$. A similar argument can be made concerning the threshold singularities amongst (3.4). It is thus plausible that as $s \to \infty$, $A_D(s,z)$ should have singularities only at $z = \pm 1$ and $z = \infty$ which represent the "condensation" of the actual physical singularities. It is yet stronger to assume that $\ln A_D$ and hence $(\alpha'_D s)^{-1}\ln A_D$ has only these singularities. (We are, of course, ignoring possible isolated zeros in $A_D$ at $t < 0$ such as nonsense, wrong singnature zeroes.) However we shall assume that, in the asymptotically leading term of $(\alpha'_D s)^{-1} \ln A_D$, the only singularities are the above branch points at $z = \pm 1$ and $\infty$. The remaining singularities which one may expect are to appear only in non leading terms. This then is assumption F as mentioned above. For technical reasons, which will be clear shortly, we cannot proceed without it (or some related analyticity assumptions) so we shall henceforth accept it.

## 4. JUSTIFICATION OF THE LOGARITHMIC SCALING LAW

We are now in a position to consider the statements B) - E) made in Section 2. Our basis for studying them is the Regge behavior assumption A) of Section 2 and the analyticity requirement F) of Section 3.

Once these are granted we can start proving the remaining statements. We first observe that because of A) the conditions of the Cerulus-Martin-Chiu-Tan[7] theorem are met individually for every quark diagram. We therefore conclude that

<u>Lemma</u> Assumption A) implies, for every quark diagram D, the bound

$$-(\alpha'_{Dt}s)^{-1}\ln|A_D(s,z)| \leq c(z)\ln s \quad \text{for fixed } z<1 \tag{4.1}$$

where c(z) is a positive definite function (of course this upper bound on $-\ln|A_D|$ is a lower bound on $|A_D|$).

To prove the remaining statements we will extensively use the method of <u>requiring consistency in the overlap region</u> between the Regge and fixed angle regions which was discussed at the outset of Section 3. As was pointed out, the Regge limit requires that eq. (2.2) hold. Now one can distinguish four cases appropriate to the fixed angle limit ($\alpha'_{Ds}$, $\alpha'_{Dt}$, $\alpha'_{Du}$ are the a priori different slopes of the dominant s, t, and u channel singularities of the diagram D):

i) $\alpha'_{Dt}\, t\ln t / \ln\beta_{Dt} \xrightarrow[|t|\to\infty]{} 0$ and similarly for $\beta_{Ds}(s)$ and $\beta_{Du}(u)$

ii) $\ln\beta_{Dt}/\alpha'_{Dt}\, t\ln t \xrightarrow[|t|\to\infty]{} 0$ and similarly for $\beta_{Ds}(s)$ and $\beta_{Du}(u)$

iii) $\ln \beta_{Dt} / \alpha'_{Dt}\, t \ln t \xrightarrow[t\to\infty]{} C_t$ and similarly for $\beta_{Ds}(s)$ and $\beta_{Du}(u)$ with at least one of the constants $C_s, C_t, C_u$ different from - 1

iv) $\ln \beta_{Dt} / \alpha'_{Dt}\, t \ln t \rightarrow -1$ and similarly for $\beta_{Ds}(s)$ and $\beta_{Du}(u)$

Consider first case i). Then, in the "overlap" region of the Regge and fixed angle regimes the ln $\beta_{Dt}(t)$ term in Eq. (2.2) will be the dominant term. If we consdier now the corresponding equation for ln $|A_D|$, it will also be dominated by the ln $|\beta_{Dt}|$ term and becomes

$$(\alpha'_{Dt}\, s)^{-1} \ln |A_D(s,t)| \xrightarrow[\text{overlap region}]{} (\alpha'_{Dt}\, s)^{-1} \ln |\beta_{Dt}(t)| \tag{4.2}$$

Note that

$$|\ln \beta_{Dt} - \ln |\beta_{Dt}|| = |\arg \beta_{Dt}(t)| \lesssim O(t) \tag{4.3}$$

holds since throughout what follows we shall be assuming linear trajectories and "signature type" phase factors. These relations, together with i), yield the following result in the overlap region

$$\frac{(1-z)\ln s}{-(\alpha'_{Dt}\, s)^{-1} \ln |A_D(s,z)|} \xrightarrow[\substack{s\to\infty \\ z \text{ fixed in overlap region}}]{} 0 \tag{4.4}$$

In particular for the t channel overlap region we are interested in, z is near 1, (recall, $t \cong -\frac{(1-z)}{2} s$ ). Similar results hold for z near - 1 (u channel) and $z \to -\infty$ (s channel). Hence for any positive definite C(z) if z is in any of the overlap regions $z \to 1$, $z \to -1$, $z \to -\infty$,

$$C(z) \ln s \underset{\substack{s\to\infty \\ z \text{ fixed}}}{<} -(\alpha'_{Dt}\, s)^{-1} \ln |A_D(s,z)| \tag{4.5}$$

which contradicts the lemma (note that we have specifically ruled out the possibility of exponentially large amplitudes i.e., we take both sides of eq. (4.1) to be positive) and this excludes case i).

For cases ii) and iii) we can now proceed by defining the function $d(z)$ as

$$-(\alpha'_{Dt}s)^{-1}\ln A_D(s,z) \xrightarrow[z = \text{fixed}]{s\to\infty} d(z)\ln s + O(1) \tag{4.6}$$

According to the lemma and our assumption of "signature type" phases this must be the leading term. (Note that it would not have been leading in case i) ). Assumption F) then implies that $d(z)$ must be an analytic function of $z$ except for possible singularities at $z = \pm 1$ and $z = \infty$.

Consider case ii). From eq. (2.2) the leading term in the (t-channel) Regge limit of $\ln A_D$ is now $\alpha'_{Dt}t \ln s$. This limit, in the overlap region, appears as $|s| \gg m_\rho^2$, $|t| \gg m_\rho^2$ but $|t/s| \ll 1$. These conditions as has already been been pointed out, then translate into $|s| \to \infty$, $z \to 1$ in the fixed angle regime. Thus we have

$$-(\alpha'_{Dt}s)^{-1}\ln A_D(s,z) \xrightarrow[z\to 1]{|s|\to\infty} -\frac{t}{s}\ln s \approx \frac{1-z}{2}\ln s \tag{4.7a}$$

or

$$d(z) \xrightarrow[z\to 1]{} \frac{1-z}{2} + O((1-z)^2) \tag{4.7b}$$

Hence $d(z)$ is regular at $z=1$. In the overlap region between the fixed $z$ and s channel Regge regimes ($|s| \gg m_\rho^2$, $|t| \gg m_\rho^2$, $|s/t| \ll 1$, or $|s| \to \infty$, $z \to -\infty$) the leading term is given by the $\alpha'_{Ds}$ s ln s part of the $\alpha'_{Ds}s \ln t$ term (the $\alpha'_{Ds}s \ln z$ part of the term is given by the next term in the asymptotic expansion in s and does not contribute to d(z)). Thus we have

$$-(\alpha'_{Dt}\, s)^{-1} \ln A_D(s,z) \xrightarrow[z \to \infty]{|s| \to \infty} -\frac{\alpha'_{Ds}}{\alpha'_{Dt}} \ln s \tag{4.8a}$$

or

$$d(z) \xrightarrow[z \to -\infty]{} -\alpha'_{Ds}/\alpha'_{Dt} \tag{4.8b}$$

Finally in the region of overlap with the u channel Regge regime ($|s| \gg m_\rho^2$, $|t| \gg m_\rho^2$, $|\frac{u}{s}| \cong |1+ t/s| \ll 1$, so that $|s| \to \infty$, $z \to -1$) the leading term is $\alpha'_{Du}\, u \ln s$ and we find

$$-(\alpha'_{Dt}\, s)^{-1} \ln A_D(s,z) \xrightarrow[z \to -1]{|s| \to \infty} -\frac{\alpha'_{Du}\, u}{\alpha'_{Dt}\, s} \ln s \simeq \frac{\alpha'_{Du}}{\alpha'_{Dt}} \left(\frac{1+z}{2}\right) \ln s \tag{4.9a}$$

or

$$d(z) \xrightarrow[z \to -1]{} \frac{\alpha'_{Du}}{\alpha'_{Dt}} \left(\frac{1+z}{2}\right) \tag{4.9b}$$

From (4.7) and (4.9) and assumption F) it follows that d(z) is an entire function of z . Condition (4.8) then implies that d(z) ≡ constant which is incompatible with (4.7) and (4.9). Thus case ii) is ruled out.

Case iii) is treated along the same lines. We find that (4.7b) - (4.9b) are replaced by

$$d(z) \xrightarrow[z \to 1]{} (1+C_t)\left(\frac{1-z}{2}\right) \tag{4.10}$$

$$d(z) \xrightarrow[z \to -\infty]{} -(1+C_s)\frac{\alpha'_{Ds}}{\alpha'_{Dt}} \tag{4.11}$$

$$d(z) \xrightarrow[z \to -1]{} (1+C_u)\frac{\alpha'_{Du}}{\alpha'_{Dt}}\left(\frac{1+z}{2}\right) \tag{4.12}$$

Assumption F) again implies that d(z) is entire and therefore that (4.10) - (4.12) are incompatible as long as at least one of $C_s$, $C_t$, $C_u$ is different from -1.

We thus find that case iv) and hence statement B), is the only remaining possibility. Let us now convince ourselves that case iv) is indeed allowed and that no contradictions arise in this case.* In case iv), $C_t = C_u = C_s = -1$ and from Eqs. (4.10) - (4.12) we see that $d(z) \equiv 0$. Thus there is now <u>no</u> ln s term in $(\alpha'_{Dt}s)^{-1}\ln A_D$. As required by the Regge regimes the leading term in $\ln A_D$ must be at least linear in s, so that

$$-(\alpha'_{Dt}s)^{-1}\ln A_D(s,z) \xrightarrow[\substack{|s|\to\infty \\ z = \text{fixed}}]{} f_D(z) + O\left(\frac{1}{\ln s}\right) \tag{4.13}$$

where $f_D(z)$ is nonzero. This establishes Statement C. It follows that it is no longer sufficient to consider only the leading asymptotic $-\alpha_{Dt}t\ln(t)$ term in $\ln\beta_{Dt}(t)$ but one must also consider terms of order t as such terms will contribute to $f_D(z)$ in (4.13). Define therefore

$$\ln\beta_{Dt}(t) \xrightarrow[|t|\to\infty]{} -\alpha'_{Dt}\,t\ln t + C_{Dt}\alpha'_{Dt}t \tag{4.14}$$

where $C_{Dt}$ is in general a complex, diagram dependent constant. The constraints which result from requiring consistency in the overlap regions (the analogues of Eqs. (4.10) - (4.12)) now become

$$f_D(z) \xrightarrow[z\to 1]{} \frac{1-z}{2}\ln\frac{2}{-(1-z)} + C_{Dt}\left(\frac{1-z}{2}\right) \tag{4.15a}$$

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*Recall that we are disregarding irrelevant complications due to signature-type factors. These will be taken up at the end of this section.

$$f_D(z) \xrightarrow[z \to -1]{} \frac{\alpha'_{Du}}{\alpha'_{Dt}} \frac{1+z}{2} \ln \frac{2}{-(1+z)} + C_{Du} \frac{\alpha'_{Du}}{\alpha'_{Dt}} \left(\frac{1+z}{2}\right) \tag{4.15b}$$

$$f_D(z) \xrightarrow[z \to -\infty]{} -\frac{\alpha'_{DS}}{\alpha'_{Dt}} \ln z - C_{DS} \frac{\alpha'_{DS}}{\alpha'_{Dt}} \tag{4.15c}$$

With the appearance of the logarithmic singularities it is useful to consider the following equivalent, but more suggestive forms:

$$f_D(z) \xrightarrow[z \to 1]{} -\frac{1-z}{2}\left[\frac{a \ln(z-1) + b \ln(1-z)}{a+b}\right] + M_{Dt}(1-z) \tag{4.16a}$$

$$f_D(z) \xrightarrow[z \to -1]{} -\frac{1+z}{2}\left[\frac{c \ln(1+z) + d \ln(-1-z)}{c+d}\right]\frac{\alpha'_{Du}}{\alpha'_{Dt}} + M_{Du}(1+z)\frac{\alpha'_{Du}}{\alpha'_{Dt}} \tag{4.16b}$$

where the M's and a, b, c, d are determined by the $C_D$'s. The logarithms are so defined that they have a cut extending from the branch point (z =+1 or - 1) to infinity along that part of the real axis where their argument is negative. On the upper lip of this cut they are defined to have an imaginary part equal to $i\pi$. These equations (4.16) then determine the nature of the singularities in $f_D(z)$ at $z = \pm 1$ and $\infty$. Using the analyticity assumption F), the nature (4.16) of the singularities at $z = \pm 1$ of $f_D(z)$ then establishes the form of $f_D(z)$, the leading term in $-(\alpha'_{Dt}s)^{-1} \ln A_D(s,z)$, to be:

$$\begin{aligned} f_D(z) = &-\frac{(1-z)}{2}\left[\frac{a \ln(1-z) + b \ln(z-1)}{a+b}\right] - \\ &-\frac{\alpha'_{Du}}{\alpha'_{Dt}}\left(\frac{1+z}{2}\right)\left[\frac{c \ln(1+z) + d \ln(-1-z)}{c+d}\right] + E(z) \end{aligned} \tag{4.17}$$

where E(z) is an entire function z. Now in the limit $z \to -\infty$ we have

$$f_D(z) \xrightarrow[z \to -\infty]{} \frac{z \ln |z|}{2}\left(1 - \frac{\alpha'_{Du}}{\alpha'_{Dt}}\right) - \frac{1}{2}\left(\frac{\alpha'_{Du}}{\alpha'_{Dt}} + 1\right) \ln |z|$$
$$+ \frac{z}{2}(i\pi)\left(\frac{b}{a+b} - \frac{\alpha'_{Du}}{\alpha'_{Dt}} \frac{c}{c+d}\right) + E_{asymptotic}(z) + O(1) \qquad (4.18)$$

Since E(z) is entire, asymptotically it cannot cancel the $z \ln |z|$ term. Such a term is ruled out by the overlap constraint (4.15c). Thus we must have

$$\alpha'_{Dt} = \alpha'_{Du} \qquad (4.19)$$

Next by comparing the $\ln |z|$ terms in (4.18) and (4.15c) and using (4.19) we find that

$$\alpha'_{Ds} = \alpha'_{Dt} \qquad (4.20)$$

Finally, we study E(z) by first considering the behavior of $f_D(z)$ in the limits $z \to -\infty$ and $z \to +\infty$. By using our knowledge of the singularity structure at infinity (given by consistency with fixed s Regge behavior, i.e. (4.15 c)) we find

$$E(z) \xrightarrow[z \to \infty]{} e - \frac{z}{2} i\pi \left(\frac{a}{a+b} - \frac{d}{c+d}\right)$$

and

$$\frac{a-b}{a+b} = \frac{d-c}{d+c}$$

where e is some constant. E(z) being entire, must have this form everywhere. Based on consistency with fixed t and fixed u Regge behavior we require $f_D(z)$ to vanish at $z = \pm 1$ (Eqs. (4.15a) and (4.15b)) and we find

$$e = \ln 2 + \frac{i\pi}{2}$$

and

$$a = b = c = d = 1/2$$

so that $E(z) = \ln z + i\pi/2$

where $+i\pi/2$ corresponds to taking the physical z values to correspond to approaching the real axis from above. Collecting all these results we find, for a diagram D with Regge behavior in all channels, i.e., no "empty" channels,

$$f_D(z) = \tilde{f}(z) = \frac{1-z}{4}\left[\ln\frac{z}{1-z} + \ln\frac{z}{z-1}\right] + \frac{1+z}{4}\left[\ln\frac{z}{1+z} + \ln\frac{z}{-1-z}\right] + \frac{i\pi}{z} \tag{4.20}$$

Except for the question of empty channels to be discussed below, this result is <u>independent</u> of the diagram D. Note that for our definitions of the logarithmic cuts $\tilde{f}(z)$ is real for all z on the upper lip of the real axis. Further, we saw that

$$\alpha'_{Dt} = \alpha'_{Ds} = \alpha'_{Du} = \alpha'_{D} \tag{4.21}$$

so that $\alpha'_D$ defines an intrinsic scale for the diagram D. We also note that the function f(z) of Eq. (4.20) differs from Veneziano's function

$$f(z) = \frac{1-z}{2}\ln\frac{z}{1-z} + \frac{1+z}{2}\ln\frac{z}{1+z} \tag{2.5}$$

only by a term that is <u>purely imaginary</u> for real z. As far as cross-sections for physical processes are concerned these two functions are completely equivalent. It is in this sense that the validity of statements D and E is to be regarded as established. We can summarize our results in the form of two theorems.

<u>Theorem 1. Assumptions A) and F) imply statements B), C), D) and E), and</u>

<u>Theorem 2. Assumptions A) and E) further imply that for any quark diagram without empty channels the trajectories of the dominant complex angular</u>

momentum plane singularities in all (three) Mandelstam channels must have the same slope (see eq. (4.21)).

This concludes our proof. What we have found is that the assumptions A) and F) are sufficient to derive all the scaling laws, stated in Section 2. We also get the bonus of Theorem 2, which, as we shall see in Section 6, leads to an interesting result concerning diffraction.

To conclude this section we shall briefly discuss the case of tree diagrams with empty channels. Consider, for example, the diagram of fig. 1a which has an empty u-channel. For $|t|$, $|s| \to \infty$ , and u fixed, this amplitude will not evidence Regge behavior (i.e., does not take the form $\beta(u)\, s^{\alpha(u)}$) but rather will fall exponentially as a function of one of the large kinematic variables, say s. In the overlap region with fixed z this requires the appearance of a negative imaginary constant term $\tilde{C}$ in the $z \to -1$ limit of $f_D(z)$. When $|s| \to \infty$ and arg s $\sim \varepsilon > 0$ (the correct Regge regime in the narrow resonance approximation) the term $- s\tilde{C}$ in $\ln A_D$ will supply the appropriate exponential fall off ($\mathrm{Re}(-s\tilde{C}) \to -\infty$). The steps given above can be easily repeated including the appropriate term of $-i\pi$ in Eqs. (4.15b). The result we find is

$$\underset{\text{empty u channel}}{f(z)} = \frac{1-z}{2} \ln \frac{2}{z-1} + \frac{1+z}{2} \ln \frac{2}{1+z} \qquad (4.22a)$$

Similarly for the other cases we find

$$\underset{\text{empty t channel}}{f(z)} = \frac{1-z}{2} \ln \frac{2}{1-z} + \frac{1+z}{2} \ln \frac{2}{-1-z} \qquad (4.22b)$$

$$\underset{\text{empty s channel}}{f(z)} = \frac{1-z}{2} \ln \frac{2}{1-z} + \frac{1+z}{2} \ln \frac{2}{1+z} \qquad (4.22c).$$

where the latter case is just the usual Veneziano result (Eq. (2.5)). Again we emphasize that the results in Eqs. (4.22) differ from Eq. (4.20) only in the imaginary part. Hence when arg s is small (the physical case for narrow resonances) and z not too close to ± 1 the corrections to logarithmic scaling due to empty channels are expected to be very small.

## 5. PHYSICAL MEANING OF THE LOGARITHMIC SCALING LAW

We have established on rather general grounds the logarithmic scaling law expressed in statements C), D), E). Of course this is not a usual scaling law connected with some form of scale-invariance. It rather tells us that the hadronic amplitude corresponding to a given quark diagram has an intrinsic scale (i.e., a very specific breaking of scale invariance) given by the common slope of the dominant Regge singularities in all its Mandelstam channels. This intrinsic scale permits the construction of an essentially unique dimensionless function of the Mandelstam variables that determines an exponential fall-off when all Mandelstam variables become large but their ratios are kept fixed. This exponential fall-off can be viewed as a simultaneous cut-off in the transverse momenta of all the (nonempty) Mandelstam channels. The high symmetry of the dimensionless function essentially shows that transverse momenta in all channels are kept within identical bounds.

To see this in more detail we remark that $f(z)$ differs from the combination

$$(\alpha'_D s)^{-1}\left[s \ln s + t \ln t + u \ln u\right] \tag{5.1}$$

only by $\pm i\pi$ which is irrelevant for the asymptotic behavior of differential cross-sections. From the form (5.1) it is clear that whatever differences there may be between the three Mandelstam channels (like different Regge intercepts, different internal quantum numbers, etc...) are "washed out" in the logarithmic scaling regime. These differences which account e.g., for forward-backward asymmetries originate in the factor $p(s,z)$ of Eq. (1.2) which is power-bounded in $s$ so that its contribution to $(\alpha'_D s)^{-1} \ln A_D$ goes to zero like $s^{-1} \ln s$ for large energies.

## 6. THE SLOPE OF THE POMERANCHUK TRAJECTORY, A SIMPLE APPLICATION OF THEOREM 2

According to the twisted loop model of diffraction,[8] the Pomeranchuk term originates in the quark diagram of fig. 1d. Many predictions of this model are in good agreement with experiment.[9] This diagram has been evaluated in dual resonance models[5,10] and it is found there, after elaborate calculations, that the slope of the Pomeranchuk trajectory is

$$\alpha'_P = \alpha'/2 \tag{6.1}$$

where $\alpha'$ is the universal slope of the hadronic ($\rho$, N, K*, $\Delta$, etc...) Regge-pole trajectories.

This result can be simply understood on the basis of Theorem 2. The Pomeranchuk in the t-channel appears as two consecutive Regge exchanges. But as these are not "flanked" by crosses (i.e., by particle-Reggeon amplitudes with third double spectral function) the resulting singularity (according to Mandelstam and Finkelstein[11]) is not a Regge-Regge cut or sum of such cuts.

By looking only in the t-channel there is thus no simple explanation of Eq. (6.1). In fact the t-channel also exhibits a double pole (renormalization effect) with slope $\alpha'$.

In the s-channel we also have two Regge exchanges, this time flanked by crosses so that they do lead to Regge-Regge (RR) cuts. These are the only and therefore dominant singularities in the s-channel. Thus

$$\alpha'_S = \alpha'_{RR} \tag{6.2}$$

But, the slope of RR-cuts is well known to be given by

$$\alpha'_{RR} = \alpha'/2 \tag{6.3}$$

By Theorem 2 we then see that the double pole in the t-channel cannot be dominant (as its slope is $\alpha'$) and that therefore the Pomeranchuk trajectory must be dominant

$$\alpha'_t = \alpha'_P \tag{6.4}$$

and

$$\frac{\alpha'}{2} = \alpha'_s = \alpha'_t = \alpha'_P \tag{6.5}$$

This proves (5.1).*

Because the same quark diagram that behaves for large $s \gg -t \gg m^2_\rho$ as $s^{\alpha_P(t)}$, behaves for $t \gg -s \gg m^2_\rho$ as $t^{\alpha_{RR}(s)}$ the slopes of the Pomeranchuk and of the RR-cut must be equal, or else the law that every quark diagram has a unique scale would be violated.

More generally, the scale of any quark diagram can be simply determined from the knowledge of the slope of the dominant trajectory in <u>any</u> one of its channels. This then requires the equality of the slopes of many further pairs of complex plane singularities. With linear Regge-trajectories many such equalities occur (e.g., $\alpha'_{RRR} = \alpha'_{PR}$, $\alpha'_{RRRR} = \alpha'_{PP}$, etc...)

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*Lovelace[12] has pointed out that the equations $\alpha'_s = \alpha'_t = \alpha'_u$ which are known to hold for the Born term of the dual resonance model, might serve as a starting point for understanding the slope of the Pomeranchuk singularity, if they could be generalized to quark diagrams with loops.

## 7. PHENOMENOLOGY OF FIXED ANGLE SCATTERING

To apply our ideas to experiment, we have to first observe that all our results pertained to the amplitudes attached to individual quark diagrams. In experiment one does not measure such amplitudes, but rather an infinite sum of quark diagram amplitudes. There is no guarantee, that such an infinite sum cannot behave in a radically different way from its individual terms.* One might consider, nevertheless, the possibility that for certain intervals in the variables s and $z = \cos\theta$ one, or a few, quark diagrams will dominate the sum. This is similar to the idea that in the Regge regime one or a few diagrams dominate the sum, which is known to work in reality. According to our logarithmic scaling law

$$\ln A_D\;(s,z) \xrightarrow[z\,=\,\text{fixed}]{|s|\to\infty} \alpha'_D\, s\; f(z) + O(\ln s) \qquad (7.1)$$

where $\alpha'_D$ is the scale of the quark diagram D. If the quark diagram D becomes dominant,

$$\ln \frac{d\sigma}{dt} \sim \ln |A|^2 \simeq \ln |A_D|^2 \qquad (7.2)$$

so that

$$\ln \frac{d\sigma}{dt} \sim 2\alpha'_D\; s\, f(z) + O(\ln s) \qquad (7.3)$$

Thus $\ln d\sigma/dt$ should be an approximately linear function of the variable $\nu = sf(z)$. In fig. 2a we present the data[13] for wide angle pp-scattering plotted against $\nu$. If we concentrate first on $\nu$ values in the range $5 \le \nu \le 10$ we note that the data at various s values seem to lie on a single

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*Although our input is very different, it could even happen that sums over infinite sets of quark diagrams exhibit the scaling behavior of Eq. (1.1).

straight line in agreement with our expectations. The slope of the straight line suggests $\alpha'_D \sim 0.4\ \mathrm{GeV}^{-2}$ for the diagram dominant in this $\mathcal{V}$ range as one would expect for single Pomeranchuk exchange. Next note that for $15 \leq \mathcal{V} \leq 30$ GeV$^2$ the data at the various s values do not agree but do seem to lie on straight lines which are parallel to a surprising degree of accuracy. This suggests considering a more general form such as in Eq. (7.3) i.e., $\ln {d\sigma}/dt + a \ln s$. In Fig. 2b we plot $\ln {d\sigma}/dt + 3.3 \ln s$ (the value 3.3 is purely phenomenological). For the range $15 \leq \mathcal{V} \leq 30$, the common slope indicates that the dominant diagram has $\alpha'_D \sim 0.26\ \mathrm{GeV}^{-2}$ which is suggestive of a PP-cut term * (as one would expect, for example, in a naive eikonal model). This combination of two terms, one dominant for $\mathcal{V} \leq 15$ and one dominant for $5 \leq \mathcal{V} \leq 30$ offers a simple explanation of the well-known break in the high energy pp-angular distribution. Although the agreement between Eq. (7.3) and the data is quite striking, it is by no means decisive.** Indeed it has been claimed that the same data are well fitted by a scaling law of the type (1.1). Thus more experiments are needed, before one can distinguish the logarithmic scaling law from asymptotic scale-invariance.

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*If this interpretation of the data in terms of P- and PP-cut exchange is correct, then one would expect $\bar{p}p$-scattering in the similar kinematical region ($5 < \mathcal{V} < 30$) to scale logarithmically just like pp-scattering. Moreover $\ln {d\sigma}/dt$ for pp and $\bar{p}p$ scattering are expected to be nearly equal in this region.

**To do a detailed fit with the two terms suggested above would require, for example, precise knowledge of the relative phases.

A most important experimental task is the measurement of large angle-high energy differential cross-sections for inelastic scattering, i.e., for reactions like $\pi N$-charge exchange scattering, associated production, or resonance production (e.g., $\pi N \rightarrow \pi\Delta$ etc...) As there is no Pomeranchuk term in such reactions and the Regge pole and cut slopes being larger than Pomeranchuk or PP-cut slopes, one expects larger values of $\alpha'_D$ for inelastic scattering than for elastic scattering.

## 8. CONCLUSIONS

We have seen that in a world with straight line Regge trajectories, individual quark diagrams have an intrinsic scale which can be obtained from the logarithmic scaling law (1.3) (or more generally (7.1))*. Dominant singularities in the complex angular momentum planes of the various (non-empty) Mandelstam channels of a given quark diagram must have the same slope as dictated by the existence of the intrinsic scale. This has led us to a simple explanation for the equality of the slopes of the Pomeranchuk singularity and of Regge-Regge cuts. Phenomenologically too, this picture agrees with the, admittedly meager, data. As was stressed before, more experiments especially on inelastic two body processes ($A+B \to A'+B'$ with $A' \neq A$ and/or $B' \neq B$) are needed before this picture can be established or ruled out.


## ACKNOWLEDGEMENTS

One of us (P. F.) takes pleasure in thanking Professor R. P. Feynman for stimulating conversations on "the rare hadronic event" of wide angle scattering. We are very grateful to Dr. Earl Swallow for his kind help with the numerical calculations leading to fig. 2, and to Dr. M. Virasoro for useful discussions.


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*It is an interesting problem to investigate logarithmic scaling laws also for exclusive multiparticle processes. Via Mueller type arguments one can then obtain logarithmic scaling laws for inclusive processes. A special case of this has been already considered by Huang and Segré and de Tar et. al.[3]

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## FIGURE CAPTIONS

Fig. 1: Quark diagrams with less than two loops for meson-meson scattering.

Fig. 2: Tests of logarithmic scaling in pp-scattering.

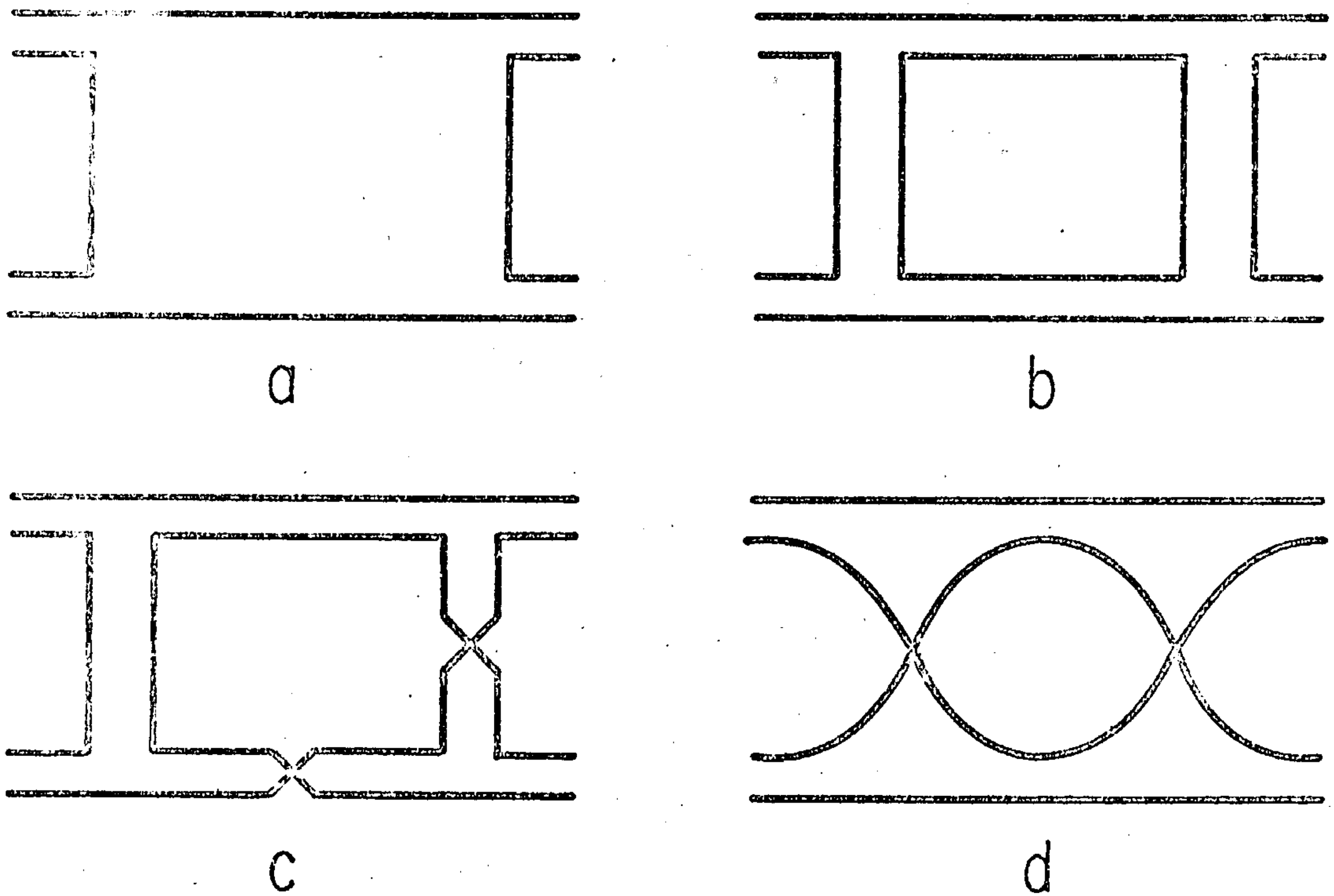



Fig.1

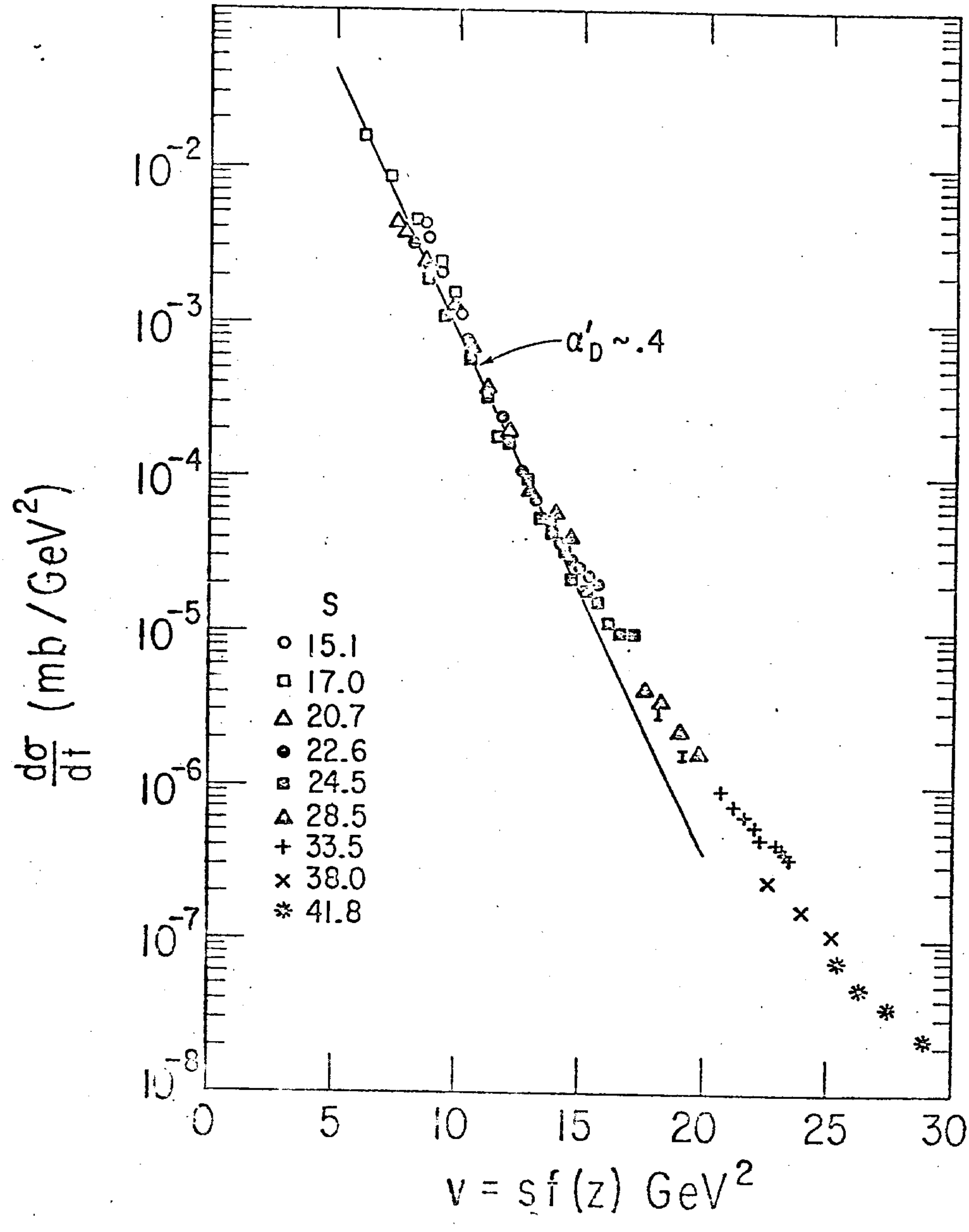
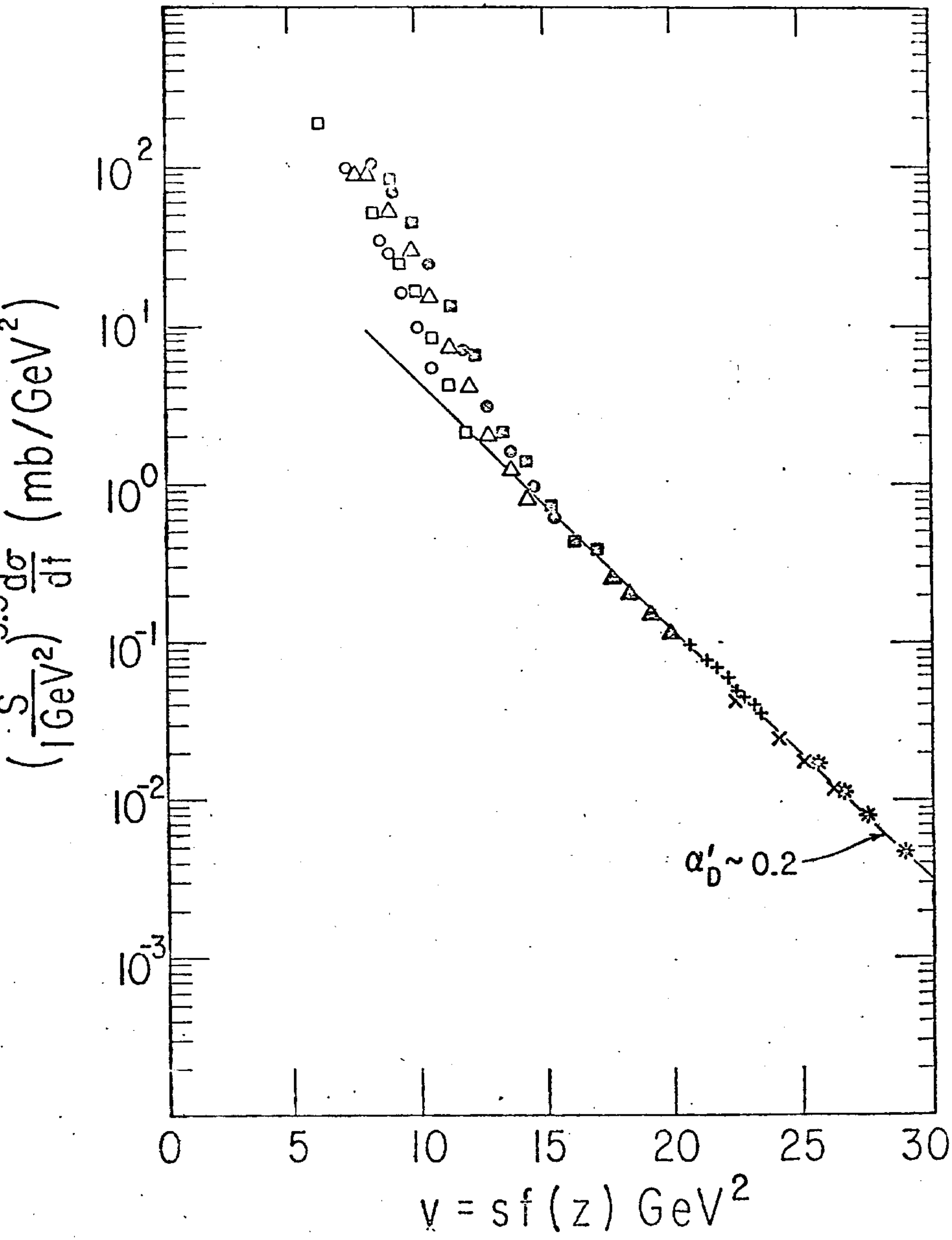

$10^{-2}$
$10^{-3}$
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
$10^{-8}$
$\frac{d\sigma}{dt}$ (mb / GeV$^2$)
$\alpha'_D \sim .4$
S
15.1
17.0
20.7
22.6
24.5
28.5
33.5
38.0
41.8
0
5
10
15
20
25
30
$\nu = s f(z)$ GeV$^2$

Fig.2(a)



$10^2$
$10^1$
$10^0$
$10^{-1}$
$10^{-2}$
$10^{-3}$
$\left(\frac{S}{1\,GeV^2}\right)^{3.3}\frac{d\sigma}{dt}$ (mb/GeV$^2$)
$\alpha'_D \sim 0.2$
0
5
10
15
20
25
30
$\nu = s f(z)$ GeV$^2$


Fig.2(b)